perties of Matter, Vol. 2, pp. 1a–39a. IFI/Plenum, New York-Washington (1970).
3. A. Degiovanni et C. Moyne, Résistance thermique de contact en régime permanent—influence de la géométrie du contact, *Rev. Gén. Therm.* **334**, 557–564 (1989).
4. M. Danis et J. Gounot, Influence de la longueur de l'échantillon sur la mesure simulée de conductivité thermique d'un matériau hétérogène, *Int. Comm. Heat Mass Transfer* **18**, 591–598 (1991).
5. A. E. Beck, A steady state method for the rapid measurement of the thermal conductivity of rocks, *J. Sci. Instrum.* **34**, 186–189 (1957).
6. E. Koutsikos, Nouvelles méthodes de détermination de la conductivité thermique des roches granitoïdes à partir de données minéralogiques et géochimiques, Thèse, Université P. et M. Curie, Paris (1985).
7. M. Danis, C. Gobbé et J. J. Royer, Analyse critique des procédures d'utilisation d'un conductivimètre à barre divisée pour des échantillons grenus: application au cas des granites, Rapport interne L.E.P.T. no. 95.2 (1995).
8. K. Hori et G. Simmons, Thermal conductivity of rock forming minerals, *Earth Planet. Sci. Lett.* **6**, 359–368 (1969).
9. J. J. Royer et M. Danis, Mise en évidence de phénomènes de réfraction thermique entre le granite de Beauvoir et son encaissant: données analytiques, *Documents BRGM* **124**, 305–310 (1988).

Pergamon

Int. J. Heat Mass Transfer. Vol. 39, No. 10, pp. 2187–2190, 1996

0017–9310/96 $15.00+0.00

0017–9310(95)00283–9

Thermal ignition in buoyancy-driven boundary layer flows along inclined hot plates

W. J. SHEU and M. C. LIN

Department of Power Mechanical Engineering, National Tsing Hua University, Hsinchu, Taiwan, Republic of China

(*Received* 22 *December* 1994 *and in final form* 6 *April* 1995)

1. INTRODUCTION

The criterion of ignition of combustible boundary-layer flow over a heated surface is a fundamental problem of combustion theory, tackled by experiment, series expansion, numerical simulation or asymptotic analysis in the limit of large activation energy. Because the behavior of most fires on earth is governed by buoyancy-induced convection, ignition in boundary-layer flow under natural convection is related not only to ignition phenomena influenced by natural convection on the academic side, but also to the onset of accidental fire on the practical side.

Laminar combustible boundary-layer flow along a vertical plate at a high-temperature wall serves as a prototypical problem to assess the effect of natural convection on ignition. Ignition phenomena of this buoyant boundary-layer flow were investigated numerically and experimentally [1, 2]. Ono *et al.* [1] proposed a numerical correlation $\ln[\sqrt{x_1}P^{n-1}] = E/RT_w + \text{constant}$ to evaluate ignition distances for selected system parameters. The results predicted from this numerical correlation agreed satisfactorily with their experimental data. However, the constant in the numerical correlation was undetermined. Obviously this constant is a function of other system parameters, e.g. reactant concentrations, *Pr* number, etc. No complete ignition criterion for such a typical problem, to the authors' knowledge, is published, even though the method of large activation energy asymptotics is well established to obtain a theoretical ignition criterion. An asymptotic analysis is superior to numerical methods because asymptotic solutions correlate all system parameters into a single equation, and thereby facilitate physical interpretations.

The ignition phenomena of combustible boundary-layer flow under purely natural convection at various angles of inclination θ are of our interest (Fig. 1). The objective of the present work is to obtain an explicit criterion of ignition, which is a function of θ, the *Pr* number, the temperature at the wall and other properties of the combustible gas, by means of an asymptotic analysis in the limit of large activation energy.

2. FORMULATION

The approximations of a steady, two-dimensional and laminar boundary-layer flow are applicable. The other assumptions are as follows: the specific heats at constant pressure of the various species are equal to a constant; radiative heat transfer, Soret and Dufour effects are neglected; the flow is the ideal gas mixture with constant values of $\rho\mu$, $\rho\lambda$ and $\rho^2 D_i$. For thermal ignition, we assume a global one-step irreversible chemical reaction ($F + \nu_O O \rightarrow$ products) with $\omega = BT^{\alpha}(\rho Y_F/W_F)^{n_F}(\rho Y_O/W_O)^{n_O}\exp(-E/RT)$ [1–3].

With the above assumptions, the conservation equations are

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \qquad (1)$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \boldsymbol{g}(\rho_\infty - \rho)\cos\theta + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \qquad (2)$$

$$\rho c_P\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = \frac{\partial}{\partial y}\left(\lambda\frac{\partial T}{\partial y}\right) + qW_F\omega \qquad (3)$$

NOMENCLATURE

A	defined in equation (27)
B	frequency factor
c	defined in equation (32)
c_p	specific heat at constant pressure
C	defined in equation (25)
D	mass diffusivity
E	activation energy
f	dimensionless stream function
$\boldsymbol{g}$	acceleration of gravity
H	$=(4\nu_\infty^2/\boldsymbol{g})^{1/3}$
n	total reaction order, n_F+n_O
n_i	reaction order of species i
P	pressure
Pr	Prandtl number
q	specific heat of combustion
R	gas constant
s	dimensionless streamwise coordinate defined in equation (9)
Sc	Schmidt number
T	temperature
T_a	dimensionless activation energy, c_pE/Rq
u	streamwise velocity
v	transverse velocity
W	molecular weight
x	coordinate along wall
y	coordinate normal to wall
Y	mass fraction.

Greek symbols

α	temperature exponent
β	$\tilde{T}_w-\tilde{T}_\infty$
Δ	defined in equation (29)
ξ	dimensionless streamwise coordinate defined in equation (18)
η	dimensionless transverse coordinate defined in equation (9)
θ	angle of inclination
λ	thermal conductivity coefficient
μ	viscosity coefficient
ν	stoichiometric coefficient
ν_∞	kinematic viscosity outside boundary layer
ζ	transformed transverse coordinate defined in equation (24)
ρ	density
ψ	stream function
ω	specific reaction rate.

Subscripts

f	frozen state
F	fuel
i	index for species
I	ignition state
O	oxidant
w	at wall
∞	outside boundary layer.

Superscripts

$\tilde{}$	dimensionless quantity
$'$	differentiation with respect to η.

$$\rho\left(u\frac{\partial Y_F}{\partial x}+v\frac{\partial Y_F}{\partial y}\right)=\frac{\partial}{\partial y}\left(\rho D_F\frac{\partial Y_F}{\partial y}\right)-W_F\omega \tag{4}$$

$$\rho\left(u\frac{\partial Y_O}{\partial x}+v\frac{\partial Y_O}{\partial y}\right)=\frac{\partial}{\partial y}\left(\rho D_O\frac{\partial Y_O}{\partial y}\right)-\nu_O W_O\omega \tag{5}$$

$$P=\rho RT \tag{6}$$

subject to the boundary conditions

$$u=v=0,\quad T=T_w,\quad \partial Y_i/\partial y=0 \quad \text{at } y=0 \tag{7}$$

$$u\to 0,\quad T\to T_\infty,\quad Y_i\to Y_{i\infty} \quad \text{as } y\to\infty. \tag{8}$$

Dimensionless coordinates and variables are introduced as follows

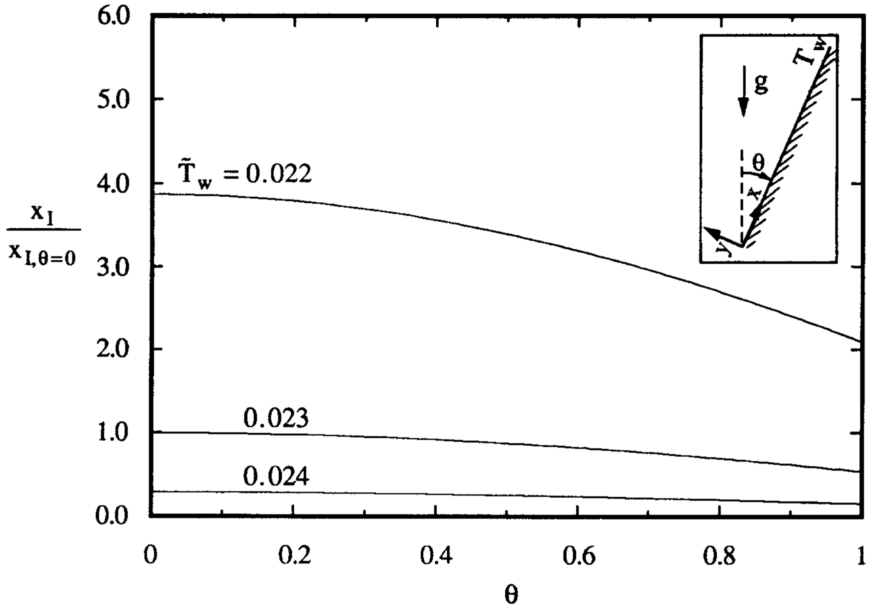

Fig. 1. $x_I/x_{I,\theta=0}$ vs angle of inclination θ (radian) for varied $\tilde{T}_w$.

$$s = x/H, \quad \eta = s^{-1/4}\int_0^{y/H} \rho(x,y')/\rho_\infty \mathrm{d}y' \tag{9}$$

and

$$\tilde{T} = c_P T/q, \quad \tilde{Y}_F = Y_F,$$

$$\tilde{Y}_O = Y_O/(v_O W_O/W_F), \quad T_a = c_P E/Rq \tag{10}$$

with $H = (4v_\infty^2/g)^{1/3}$. The dimensionless stream function f is defined as

$$f(s,\eta) = \psi(x,y)/4v_\infty s^{3/4}, \tag{11}$$

where ψ satisfies

$$\partial\psi/\partial x = -\rho v/\rho_\infty \quad \text{and} \quad \partial\psi/\partial y = \rho u/\rho_\infty. \tag{12}$$

Let $f' \equiv \partial f/\partial\eta$, $Pr \equiv Pr_\infty$ and $Sc_i \equiv Sc_{i\infty}$. The appropriate dimensionless equations are

$$\frac{\partial^2 f'}{\partial\eta^2} + 3f\frac{\partial f'}{\partial\eta} + 2\zeta\left(\frac{\partial f}{\partial\zeta}\frac{\partial f'}{\partial\eta} - f'\frac{\partial f'}{\partial\zeta}\right) = 2f'^2 + \left(1 - \frac{\tilde{T}}{\tilde{T}_\infty}\right)\cos\theta \tag{13}$$

$$Pr^{-1}\frac{\partial^2\tilde{T}}{\partial\eta^2} + 3f\frac{\partial\tilde{T}}{\partial\eta} + 2\zeta\left(\frac{\partial f}{\partial\zeta}\frac{\partial\tilde{T}}{\partial\eta} - f'\frac{\partial\tilde{T}}{\partial\zeta}\right) = -\zeta\tilde{\omega} \tag{14}$$

$$Sc_i^{-1}\frac{\partial^2\tilde{Y}_i}{\partial\eta^2} + 3f\frac{\partial\tilde{Y}_i}{\partial\eta} + 2\zeta\left(\frac{\partial f}{\partial\zeta}\frac{\partial\tilde{Y}_i}{\partial\eta} - f'\frac{\partial\tilde{Y}_i}{\partial\zeta}\right) = \zeta\tilde{\omega}, \quad i = F,O \tag{15}$$

associated with the boundary conditions

$$f' = 0, \quad v = 0, \quad \tilde{T} = \tilde{T}_w, \quad \partial\tilde{Y}_i/\partial\eta = 0 \quad \text{at} \quad \eta = 0 \tag{16}$$

$$f' \to 0, \quad \tilde{T} \to \tilde{T}_\infty, \quad \tilde{Y}_i \to \tilde{Y}_{i\infty} \quad \text{as} \quad \eta \to \infty, \tag{17}$$

where the dimensionless streamwise coordinate

$$\zeta = 2B(x/g)^{1/2}(P/R)^{n-1}v_O^{n_O}W_F^{1-n}(q/c_P)^{1+\alpha-n} \tag{18}$$

and the reaction rate

$$\tilde{\omega} = \tilde{T}^{1+\alpha-n}\tilde{Y}_F^{n_F}\tilde{Y}_O^{n_O}\exp(-T_a/\tilde{T}). \tag{19}$$

3. IGNITION ANALYSIS

As a standard procedure of asymptotic analysis [3], we seek first the solutions in the chemically frozen state. In the absence of a chemical reaction as $T_a \to \infty$, this problem admits similar solutions that depend on η only. Thus we have

$$f''' + 3ff'' = 2f'^2 + (1 - \tilde{T}/\tilde{T}_\infty)\cos\theta \tag{20}$$

$$Pr^{-1}\tilde{T}'' + 3f\tilde{T}' = 0 \tag{21}$$

$$Sc_i^{-1}\tilde{Y}_i'' + 3f\tilde{Y}_i' = 0, \quad i = F,O \tag{22}$$

with boundary conditions (16) and (17). Moreover, the boundary condition $v = 0$ at $\eta = 0$ (16) can be reduced to $f = 0$ at $\eta = 0$. This is a standard two-point boundary-value problem that is solved numerically using a computer-library subroutine (BVPFD from IMSL User's Manual, 1989) [4]. In particular, the solutions of species are simply

$$\tilde{Y}_{i,f} = \tilde{Y}_{i\infty}, \quad i = F,O. \tag{23}$$

The location of ignition is expected to be near the wall because the temperature there is the greatest. In the limit of large activation energy, the variation of temperature from the chemically frozen state to the weakly reactive state at ignition is expected to be quite small $[O(\varepsilon)]$ [3]. As a result, the stream function f and zeroth-order concentrations of reactants are assumed to be the same as those in the chemically frozen state, when we solve the energy equation with a weakly chemical reaction term.

A new coordinate transformation is introduced as

$$\xi = \int_0^\eta \exp\left(-3Pr\int_0^{\eta'} f(\eta'')\,\mathrm{d}\eta''\right)\mathrm{d}\eta'/C \tag{24}$$

with

$$C(Pr,\theta,\tilde{T}_w,\tilde{T}_\infty) = \int_0^\infty \exp\left(-3Pr\int_0^{\eta'} f(\eta'')\,\mathrm{d}\eta''\right)\mathrm{d}\eta' \tag{25}$$

In terms of this new transverse coordinate, equation (14) is rewritten as

$$\frac{\partial^2\tilde{T}}{\partial\xi^2} - 2\zeta f' Pr A(\eta)\frac{\partial\tilde{T}}{\partial\zeta} = -\zeta\tilde{\omega} Pr A(\eta), \tag{26}$$

where

$$A(\eta) = \left(C\exp\left(3Pr\int_0^\eta f(\eta')\,\mathrm{d}\eta'\right)\right)^2 \tag{27}$$

and the solution of temperature in the chemically frozen limit becomes recast into

$$\tilde{T}_f = \tilde{T}_w - \beta\xi \tag{28}$$

with $\beta = \tilde{T}_w - \tilde{T}_\infty$.

Equation (26) is similar to that of the previous work [3]. The detailed procedure of asymptotic analysis can be found and is not reproduced here. The key steps, which are quite standard in the asymptotic analysis of ignition, are illustrated as follows. The solution in the inner diffusive–reactive region near the wall is locally similar after introducing an inner expansion and a stretched coordinate. Besides the isothermal condition at the wall, the second boundary condition is obtained through matching with the outer diffusive-convective solution which is chemically frozen to all orders. A relevant Damköhler number is defined as

$$\Delta = 2\varepsilon\, Pr\, C^2\beta^{-2}\zeta_I\tilde{T}_w^{1+\alpha-n}\tilde{Y}_{F\infty}^{n_F}\tilde{Y}_{O\infty}^{n_O}\exp(-T_a/\tilde{T}_w) \tag{29}$$

with a small expansion parameter $\varepsilon = \tilde{T}_w^2/\theta$. Finally, this problem admits double solutions for $\Delta < 1$ but no solutions for $\Delta > 1$; thereby the Damköhler at ignition is $\Delta_I = 1$ [3].

According to equation (29), the corresponding dimensional ignition distance is

$$x_I = \left[\left(\frac{g^{1/2}\beta^2}{4\varepsilon BPrC^2}\right)\left(\frac{W_F^{n_F-1}W_O^{n_O}}{Y_{F\infty}^{n_F}Y_{O\infty}^{n_O}}\right) \times \left(\frac{P}{R}\right)^{1-n} T_w^{n-\alpha-1}\exp\left(\frac{E}{RT_w}\right)\right]^2. \tag{30}$$

4. RESULTS AND CONCLUSIONS

According to the above equation, the ignition distance x_1 becomes recast into

$$\ln[\sqrt{x_I}P^{n-1}] = E/RT_w + c \tag{31}$$

with the constant c as

$$c = \ln\left[\left(\frac{g^{1/2}\beta^2}{4\varepsilon B\, Pr\, C^2}\right)\left(\frac{W_F^{n_F-1}W_O^{n_O}}{Y_{F\infty}^{n_F}Y_{O\infty}^{n_O}}\right)R^{n-1}T_w^{n-\alpha-1}\right]. \tag{32}$$

Equation (31) is identical with the experimentally verified correlation proposed by Ono *et al.* [1] for a vertical plate. Here we not only analytically determine this constant c, but also extend the application of equation (31) to all inclined plates.

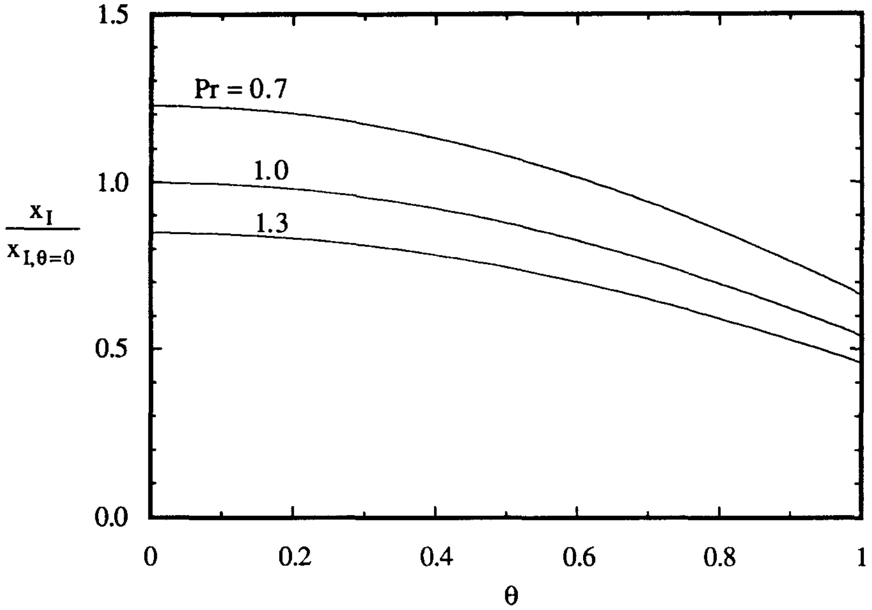

Fig. 2. $x_I/x_{I,\theta=0}$ vs angle of inclination θ (radian) for varied Pr number.

For quantitative presentations, we adopted the input system parameters $n_F = 0.15$, $n_O = 1.65$, $\alpha = 0.0$, $\tilde{T}_\infty = 0.0075$, $T_a = 0.4$ and $\tilde{Y}_{F\infty} = \tilde{Y}_{O\infty} = 0.06$. A plot of $x_I/x_{I,\theta=0}$ vs θ for varied $\tilde{T}_w$ is presented in Fig. 1. Here $x_{I,\theta=0}$ is the ignition distance for a vertical plate ($Pr = 1$ and $\tilde{T}_w = 0.023$). Obviously a slightly increased wall temperature results in a substantially decreased ignition distance for a large activation energy. For fixed $\tilde{T}_w$, the magnitude of $x_I/x_{I,\theta=0}$ decreases with increasing angle of inclination θ. This fact signifies that a weaker streamwise flow is induced for a greater value of θ. The value of $x_I/x_{I,\theta=0}$ vs θ for some selected Pr numbers is shown in Fig. 2. According to this figure, the ignition distance for a larger Pr number is smaller than that for a smaller Pr number. The Pr number is defined as a ratio of momentum diffusivity to thermal diffusivity. From the physical point of view, the thermal energy generated from the chemical reaction near the high-temperature wall is transported away from it at a smaller rate for a greater Pr number; thereby the ignition criterion is satisfied more readily and then the magnitude of ignition distance decreases.

The dependence of the ignition distance (x_I) on the acceleration of gravity ($\boldsymbol{g}$), the ambient pressure (P) and the concentrations of reactants ($Y_{F\infty}$ and $Y_{O\infty}$) can be readily determined from equation (30). The result shows $x_I \sim \boldsymbol{g}P^{2(1-n)}Y_{F\infty}^{2n_F}Y_{O\infty}^{2n_O}$. For common hydrocarbon fuels, the total reaction order n exceeds unity [5]. Thus the growth of the ambient pressure results in decreasing x_I.

REFERENCES

1. S. Ono, H. Kawano, H. Niho and G. Fukuyama, Ignition in a free convection from vertical hot plate, *Bull. JSME* **19**, 676–683 (1976).
2. L. D. Chen and G. M. Faeth, Ignition of a combustible gas near heated vertical surfaces, *Combust. Flame* **42**, 77–92 (1981).
3. C. K. Law and H. K. Law, Thermal-ignition analysis in boundary-layer flows, *J. Fluid Mech.* **92**, 97–108 (1979).
4. Fortran Subroutine for Mathematical Applications, *IMSL Math/Library*, Version 1.1 (1989).
5. C. K. Westbrook and F. L. Dryer, Simplified reaction mechanisms for the oxidation of hydrocarbon fuels in flames, *Combust. Sci. Tech.* **27**, 31–42 (1981).